

Central Extension of a New W_∞ -Type Algebra

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Abstract. The central extension of a new infinite dimensional algebra which has both W_∞ and affine $sl(2, R)$ as subalgebras is found. The critical dimension of the corresponding string model is $D = 5$.

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Recently we obtained [1] a new W_∞ -type algebra as a dynamical symmetry algebra of the so called ‘generalized Chern-Simons string’. In this letter our goal is to find the possible nontrivial central extension of this algebra and the critical dimension of the model.

The generalized Chern-Simons string is characterized by the following constraint system

$$P^\mu P_\mu^{(n)} \approx 0, \quad X^\mu X_\mu^{(n)} \approx 0, \quad P^\mu X_\mu^{(n)} \approx 0, \quad n = 0, 1, 2, \dots, \quad (1)$$

where $X_\mu(\tau, \sigma)$ give the embedding of the string into a D -dimensional Minkovski spacetime and P_μ are the canonically conjugated to X_μ momenta. The notations $X_\mu^{(n)}$ and $P_\mu^{(n)}$ used in (1) and bellow stand for the derivatives $\partial^n / \partial \sigma^n (X_\mu)$ and $\partial^n / \partial \sigma^n (P_\mu)$. Using the basic commutation relations between coordinate and momentum it is easy to find the equal time commutation relations between constraints (1).

In the present letter instead of (1) we shall use another basis of generators. There are two reasons for this. First, as it is easy to see, the generators (1) are not independent (e.g. P^2 and $PP^{(1)}$). Second, it turns out that (1) is not quite suitable for the estimation of the central extension. The basis we shall use is

$$\frac{(P^{(n)})^2}{2}, \quad \frac{(X^{(n)})^2}{2}, \quad PX^{(n)}, \quad n = 0, 1, 2, \dots \quad (2)$$

It is convenient to introduce the functional representation of the above generators

$$\begin{aligned} \mathcal{P}_l[f] &= \int d\sigma f(\sigma) \frac{(P^{(l)})^2}{2} \\ \mathcal{X}_m[g] &= \int d\sigma g(\sigma) \frac{(X^{(m)})^2}{2} \\ \mathcal{G}_n[h] &= \int d\sigma h(\sigma) PX^{(n)}. \end{aligned} \quad (3)$$

Here $f(\sigma), g(\sigma)$ and $h(\sigma)$ are arbitrary functions on the circle (we consider a closed string only).

In order to establish the commutation relations in the basis (2) and to find how to pass from this basis to the old one (the inverse transition from (1) to (2) is trivial) one has to learn to express quantities like $\mathcal{F}_{n,m}[g] = \int d\sigma g(\sigma) P_\mu^{(n)} P^{\mu(n+m)}$ in terms of $\mathcal{P}_k[g]$. Using the identity

$$\mathcal{F}_{n,m}[g] = -\mathcal{F}_{n,m-1}[g^{(1)}] - \mathcal{F}_{n+1,m-2}[g]$$

and the initial condition $\mathcal{F}_{n,1}[g] = -\mathcal{P}_n[g^{(1)}]$ one can prove that

$$\mathcal{F}_{n,m}[g] = \sum_k^{\lfloor \frac{m}{2} \rfloor} (-1)^{m+k} \frac{m}{m-k} \binom{m-k}{k} \mathcal{P}_{n+k}[g^{(m-2k)}], \quad m > 0, \quad (4)$$

where $\lfloor m/2 \rfloor$ is the greatest integer less than or equal to $m/2$. Using formula (4) and a similar one for $\int d\sigma f(\sigma) X_\mu^{(n)} X^{\mu(n+m)}$ we obtain the following (equal time) commutation relations between generators (2) (only the nontrivial ones are listed)

$$[\mathcal{G}_m[f], \mathcal{G}_n[g]] = \sum_{k=0}^n \binom{n}{k} \mathcal{G}_{m+k}[f^{n-k}g] - \sum_{k=0}^m \binom{m}{k} \mathcal{G}_{n+k}[fg^{m-k}] \quad (5)$$

$$[\mathcal{P}_m[f], \mathcal{G}_n[g]] = \sum_{k=0}^{m+n} \sum_{l=0}^{\lfloor \frac{m+k}{2} \rfloor} (-1)^{k+l} \binom{m+n}{k} \binom{m+k-l}{l} \frac{m+k}{m+k-l} \times \mathcal{P}_l[(gf^{(m+n-k)})^{(m+k-2l)}] \quad (6)$$

$$[\mathcal{X}_m[f], \mathcal{G}_n[g]] = \sum_{k=0}^m \sum_{l=0}^{\lfloor \frac{a+k}{2} \rfloor} (-1)^{n+k+l+1} \binom{m}{k} \binom{a+k-l}{l} \frac{a+k}{a+k+l} \times \mathcal{X}_{b+l}[h_{m-k}^{(a+k-2l)}] \quad (7)$$

$$[\mathcal{P}_m[f], \mathcal{X}_n[g]] = (-1)^{m+n} \sum_{k=0}^{m+n} \sum_{l=0}^m \binom{m+n}{k} \binom{m}{l} \times \mathcal{G}_{n+k+l}[(fg^{(m+n-k)})^{(m-l)}] \quad (8)$$

In eq.(7) $a = |m - n|$, $b = \max\{m, n\}$, $h_{m-k} = gf^{(m-k)}$ for $m \geq n$ and $h_{m-k} = fg^{(m-k)}$ for $n > m$.

One immediately recognizes two important subalgebras of the above algebra — the first one is $sl(2, R)$ with generators $P^2/2, X^2/2, PX$, and the second is $DOP(S^1)$ with generators $PX^{(n)}$, $n = 0, 1, 2, \dots$. Both these algebras have unique nontrivial central extensions. The corresponding two-cocycles are

$$c(\mathcal{G}_m[f], \mathcal{G}_n[g]) = c \frac{m!n!}{(m+n+1)!} \int d\sigma f^{(n)} g^{(m+1)} \quad (9)$$

for $DOP(S^1)$ [2] and

$$c(\mathcal{G}_0[f], \mathcal{G}_0[g]) = -2c(\mathcal{P}_0[f], \mathcal{X}_0[g]) = k \int d\sigma f g^{(1)} \quad (10)$$

for $\hat{sl}(2, R)$ [3] (in the Chevalley basis used here the $sl(2, R)$ Killing metric is proportional to the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and this determines the “index” structure of (10)). It is clear from eqs. (9, 10) that the constants k and c have to be equal. Moreover, a nontrivial two-cocycles $c(\mathcal{P}_m[f], \mathcal{X}_n[g]) = -c(\mathcal{X}_n[g], \mathcal{P}_m[f])$ (which generalize the second one in (10)) have to exist for any m and n in order for the whole algebra to possess a central extension. Using the Jacoby identity

$$c([\mathcal{G}_l[f], \mathcal{X}_m[g]], \mathcal{P}_n[h]) + c([\mathcal{X}_m[g], \mathcal{P}_n[h]], \mathcal{G}_l[f]) + c([\mathcal{P}_n[h], \mathcal{G}_l[f]], \mathcal{X}_m[g]) = 0 \quad (11)$$

(there are no other nontrivial Jacoby identities except (11) and the one which involves only \mathcal{G}_m and determines (9)) one can prove that

$$c(\mathcal{P}_m[f], \mathcal{X}_n[g]) = -\frac{c}{2} \frac{(m+n)!(m+n)!}{(2m+2n+1)!} \int d\sigma f^{(2n)} g^{(2m+1)} \quad (12)$$

The cocycles (9), (12) determine the unique nontrivial central extension of the algebra in consideration. The following formulas are used to obtain them

$$\sum_{k=0}^n (-1)^k \frac{(m+k)!}{(m+p+k)!} \binom{n}{k} = \frac{m!(n+p-1)!}{(p-1)!(m+n+p)!}, \quad \text{for } p \geq 1 \quad (13)$$

$$\sum_{k=0}^n (-1)^k \frac{(m+p+k)!}{(m+k)!} \binom{n}{k} = 0, \quad \text{for } p < n \quad (14)$$

$$\sum_{k=0}^n (-1)^k \frac{(m+p+k)!}{(m+k)!} \binom{n}{k} = (-1)^n \frac{p!(m+p)!}{(p-n)!(m+n)!}, \quad \text{for } p \geq n \quad (15)$$

$$\sum_{k=0}^n (-1)^k \frac{p}{p-k} \frac{(m+k)!^2}{(2m+2k+1)!} \binom{p-k}{k} = 2 \frac{m!(m+p)!}{(2m+p+1)!},$$

for $p = 2n-1, 2n, 2n+1$ (16)

The first three of these identities can be found in [2] (see also [4]). One can prove for example (13) first for $p = 1$ using the residue theorem for the function $1/(z \prod_{i=m+1}^{m+n+1} (z-i))$ and then by induction for any p . The second and the third formula ((14) and (15)) can be shown analogously. Formula (16) can be proven by induction following the chainlet (here $M(n, p)$ denotes identity (16))

$$\begin{aligned} M(0, 1) &\rightarrow \dots \rightarrow M(n, 2n-1) \rightarrow M(n, 2n) \rightarrow \\ &\rightarrow M(n, 2n+1) \rightarrow M(n+1, 2n+1) \rightarrow \dots \end{aligned}$$

Finally, we want to make some comments about the critical dimension of our model. A natural way to find it is to consider the corresponding BRST anomaly as in the standard string theory [5]. This is however a complicated task due to the infinite number of generators (2), but fortunately, there is no need to solve it completely. The reason is very simple and to explain it we need only the basic formulas of the BRST quantization procedure: For a gauge algebra with generators J_i and structure constants f_{ij}^k the BRST charge is [6]

$$Q = c^i J_i - \frac{1}{2} f_{ij}^k c^i c^j b_k \quad (17)$$

where $\{c^i, b_i\}$ are the ghost-antighost pairs corresponding to the generators J_i . The total symmetry generators (with the ghost contributions) are

$$J_i^{tot} = J_i + J_i^{gh} = [Q, b_i]_+ = J_i - f_{ij}^k c^j b_k \quad (18)$$

The generators $J_i^{gh} = -f_{ij}^k c^j b_k$ satisfy the same algebra as J_i and therefore the central extension for them up to a multiplicative constant is the same as for J_i . (The central extension in both cases is equal to the vacuum expectation value of the corresponding commutator.) For our algebra there is only one unfixed constant in the possible central extension and so, the cancellation of the anomaly in one commutator will lead automatically to the cancellation of the entire anomaly in all commutators. This fact allows us to consider only one commutator and in what follows we shall concentrate our attention on the anomaly in \mathcal{G}_1 — \mathcal{G}_1 commutator (Virasoro subalgebra).

A short note has to be added at this point. It concerns our choice of the commutator in which we estimate the anomaly. At a first sight the simplest one is not the commutator we plan to investigate but the \mathcal{G}_0 — \mathcal{G}_0 one. However, since \mathcal{G}_0 forms an affine $U(1)$ algebra there are problems with the definition of the BRST charge. Our experience shows [7] that in this case a modification of the BRST procedure is needed but this question is far away from our present goal and we shall not deal with it here.

Bellow we use the notations $\{c_m^{\mathcal{G}}, b_m^{\mathcal{G}}\}, \{c_m^{\mathcal{X}}, b_m^{\mathcal{X}}\}, \{c_m^{\mathcal{P}}, b_m^{\mathcal{P}}\}$ for the ghost-antighost pairs corresponding to the constraints $\mathcal{G}_m, \mathcal{X}_m$ and \mathcal{P}_m respectively. We use also the notations $p_i^\mu, x_i^\mu, c_{m,i}^{\mathcal{G}}, b_{m,i}^{\mathcal{G}}$ and so on for the Fourier components of $P^\mu, X^\mu, c_m^{\mathcal{G}}, b_m^{\mathcal{G}}$ and so on. Using eqs.(5–8) and (17) it is easy to obtain the BRST charge for our algebra and to select the terms which determine \mathcal{G}_1^{gh} . From them only the following ones contribute to the Virasoro anomaly

$$\begin{aligned} & \sum_{m=0}^{\infty} \int [m c_m^{\mathcal{G}} c_1^{\mathcal{G}(1)} - c_1^{\mathcal{G}} c_m^{\mathcal{G}(1)}] b_m^{\mathcal{G}} \\ & + \sum_{m=0}^{\infty} \int [(2m+1) c_m^{\mathcal{P}} c_1^{\mathcal{G}(1)} + c_1^{\mathcal{G}} c_m^{\mathcal{P}(1)}] b_m^{\mathcal{P}} \\ & + \sum_{m=0}^{\infty} \int [(2m-1) c_m^{\mathcal{X}} c_1^{\mathcal{G}(1)} + c_1^{\mathcal{G}} c_m^{\mathcal{X}(1)}] b_m^{\mathcal{X}} \end{aligned}$$

According to (18)

$$\begin{aligned} \mathcal{G}_{1,i}^{gh} = \dots & + \sum_{m=0}^{\infty} \sum_{j=-\infty}^{\infty} (j-mi) c_{m,-j}^{\mathcal{G}} b_{m,i+j}^{\mathcal{G}} \\ & + \sum_{m=0}^{\infty} \sum_{j=-\infty}^{\infty} (j-(2m+1)i) c_{m,-j}^{\mathcal{P}} b_{m,i+j}^{\mathcal{P}} \\ & + \sum_{m=0}^{\infty} \sum_{j=-\infty}^{\infty} (j-(2m-1)i) c_{m,-j}^{\mathcal{X}} b_{m,i+j}^{\mathcal{X}} \end{aligned}$$

where dots (\dots) stands for the terms which do not contribute to the anomaly. Using a standard ghost vacuum for the anomaly we get

$$\begin{aligned} \langle 0 | [\mathcal{G}_{1,i}^{gh}, \mathcal{G}_{1,i'}^{gh}] | 0 \rangle & = -\delta_{i+i'} \sum_{m=0}^{\infty} \left[(9m^2 + 5m + \frac{5}{2}) i^3 - \frac{1}{2} i \right] \\ & = -\delta_{i+i'} \left[\sum_{m=1}^{\infty} \left[(9m^2 + 5m + \frac{5}{2}) i^3 - \frac{1}{2} i \right] + \frac{5}{2} i^3 - \frac{1}{2} i \right] \\ & = -\delta_{i+i'} \left[\frac{5}{6} i^3 - \frac{1}{4} i \right] \end{aligned} \tag{19}$$

To obtain the last line in the above equation we use the ζ -function regularization of the sums. The estimation of the matter field anomaly is simpler. Skipping the details, the answer we find is

$$\langle 0 | [\mathcal{G}_{1,i}, \mathcal{G}_{1,i'}] | 0 \rangle = \delta_{i+i'} \left[\frac{D}{6} i^3 - \frac{D}{6} i - 2i\beta \right] \tag{20}$$

where β is the normal ordering ambiguity constant in \mathcal{G}_1 . From eqs.(19, 20) we see that the model is anomaly free if

$$D = 5, \quad \beta = -\frac{7}{24} \quad (21)$$

The obtained critical dimension is “more physical” compared to that of the pure W_∞ string — The contribution of the $DOP(S^1)$ ghosts to the anomaly (19) is

$$-\delta_{i+i'} \sum_{m=0}^{\infty} \left[(m^2 + m + \frac{1}{6})i^3 - \frac{1}{6}i \right].$$

Proceeding as above one can find that the critical dimension of the pure $W_{1+\infty}$ string (model with only $DOP(S^1)$ constraints) is

$$D = 0$$

and that of the W_∞ string (model without spin one generator \mathcal{G}_0) is

$$D = -1.$$

The latter value differs from that obtained earlier [8] for the W_∞ string which is $D = -2$. The difference is due to the realization of the (matter field) generators \mathcal{G} used here. As a consequence of this realization if we consider only the pure Virasoro subalgebra we shall get $D = 13$ instead of $D = 26$. A final note: All remarks in [8] concerning the arbitrariness in the determination of D are also valid here.

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